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On the directional approach in constitutive modelling: A general thermomechanical framework and exact solutions for Mooney–Rivlin type elasticity in each direction

Alexander Lion [a,*], Nico Diercks [a], Julien Caillard [b]

[a] *Institute of Mechanics, Universität der Bundeswehr München, Neubiberg, Germany*
[b] *Manufacture Française des Pneumatiques Michelin, Clermont-Ferrand, France*

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ABSTRACT

In order to represent process-induced anisotropies in continuum mechanics or to transfer one-dimensional material models to three spatial dimensions the directional approach is a helpful technique. Since the essential equations are defined in the orientation space it is also denoted as microsphere approach. In the current article, the relation for the directional stress tensor of the second Piola–Kirchhoff type is motivated using the volumetric/isochoric split of the deformation gradient and the Clausius–Duhem inequality. Owing to inherent nonlinearities, numerical discretisation techniques are usually applied to calculate the total stress by averaging the directional stress tensors over the unit sphere. In order to investigate the accuracy of such simulations, the availability of exact solutions in closed form is essential. To this end, the tension/compression behaviour which belongs to a certain direction in the orientation space is modelled by an elasticity relation of the Mooney Rivlin type. The exact solutions are calculated, visualized and discussed for uniaxial tension and compression as well as for equibiaxial tension.

1. Introduction

In order to develop tensor-valued constitutive relations for finite deformations and elastic or inelastic material behaviour, there are different possibilities in continuum mechanics. In this context, the reader is referred to the textbooks written by Haupt (2002), Truesdell and Noll (1992), Krawietz (1986) or Malvern (1969). In conventional approaches, the second Piola–Kirchhoff stress tensor or the specific free energy is represented as a functional of the Green strain tensor and the temperature. A relatively new approach which is the starting point of this article has been introduced and applied by Freund and Ihlemann (2010), Shutov et al. (2011), Freund et al. (2011) or Naumann and Ihlemann (2012). It is capable to represent deformation-induced anisotropies and has similarities with the microsphere approach which was developed one decade earlier by Pawelski (1998). In the meantime, the microsphere or microplane approach has been formulated for large as well as for infinitesimal deformations. It is associated to the material behaviour on the microscale and has been applied to polymers, metals or biological materials (cf. Carol et al. (2004), Miehe et al. (2004), Göktepe and Miehe (2005, 2008), Miehe and Göktepe (2005), Menzel and Waffenschmidt (2009), Ostwald et al. (2010) or Waffenschmidt and Menzel (2012)). Since the constitutive representation of the elastic behaviour of many polymers becomes easier when non-affine relations between the macroscopic stretches and those which act on the chain molecules are assumed, Miehe et al. (2004) considered this in their extended microsphere model. In this approach, a so-called p-root averaging operator is used which introduces an additional nonlinearity. Such effects are not taken into account in the current consideration because it is addressed to the derivation of exact solutions in closed form without any approximation of the averaging operator. In Freund and Ihlemann (2010), no interpretation in terms of a micro-macro transition is provided and the physical inspiration is different. The essential idea is the superposition of a continuous distribution of one-dimensional stress-strain relations to build the second Piola–Kirchhoff stress tensor. This procedure is carried out in the orientation space of the reference configuration. The one-dimensional stress-strain models must not necessarily be related to the microstructure of the material. In Freund and Ihlemann (2010), the central idea has been worked out and exemplified. Naumann and Ihlemann (2012) have demonstrated that the superposition of thermodynamical consistent one-dimensional models leads to a three-dimensional model that is also thermodynamically consistent. In the current article, the inspiration of Freund and Ihlemann (2010) is pursued.

* Corresponding author. Tel.: +49 0 89 60042494; fax: +49 0 89 60042386.
E-mail address: alexander.lion@unibw.de (A. Lion).

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In order to evaluate numerical integration schemes for the angular averaging operator or to understand the behaviour of the directional approach in more detail, exact solutions in closed form are supportive. In this context, Carol et al. (2004) formulated appropriate microplane models which lead, after the angular integration, to macroscopic stress-strain relations of the Neo-Hookean and the Mooney–Rivlin type. In the current article, the opposite is done: in the directional space, Mooney–Rivlin type behaviour is assumed, the angular averaging operator is evaluated without any approximation and the resulting macroscopic behaviour is calculated in closed form. To the knowledge of the authors, this has not been done in the literature.

The present article is structured as follows. In the next section, some basics are provided. In Section 3, a thermodynamic investigation is realised and in Section 4, fundamental properties of the directional approach are analysed. In Section 5, the exact solutions for a special type of nonlinear elastic material behaviour of the Mooney–Rivlin type under uniaxial tension and compression as well as under equibiaxial tension are derived, visualised and discussed.

2. Fundamentals

At first, some basics of continuum mechanics are introduced (cf. Truesdell and Noll (1992) or Haupt (2002)). In order to separate changes in volume from changes in shape the deformation gradient $\mathbf{F}$ is multiplicatively decomposed in the standard way:

$$\mathbf{F} = \overline{\mathbf{F}}\hat{\mathbf{F}}, \quad \overline{\mathbf{F}} = J^{1/3}\mathbf{1}, \hat{\mathbf{F}} = J^{-1/3}\mathbf{F}, \quad J = \det \mathbf{F} \tag{1}$$

Now, the right Cauchy Green tensor $\mathbf{C}$, its determinant III = det $\mathbf{C} = J^2$, its isochoric part $\hat{\mathbf{C}}$ and the Green strain tensor $\mathbf{E}$ are introduced:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} \tag{2}$$

$$\hat{\mathbf{C}} = \hat{\mathbf{F}}^{\mathrm{T}}\hat{\mathbf{F}} = \mathrm{III}^{-1/3}\mathbf{C} \tag{3}$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}) \tag{4}$$

In order to calculate physical quantities in given directions which are defined by the angles ϑ and φ a unit vector is defined in the reference configuration as sketched in Fig. 1:

$$\overset{\alpha}{\vec{e}}(\vartheta, \varphi) = \sin\vartheta\cos\varphi\,\vec{e}_1 + \sin\vartheta\sin\varphi\,\vec{e}_2 + \cos\vartheta\,\vec{e}_3 \tag{5}$$

If the deformation gradient or the right Cauchy Green tensor is known, the isochoric stretch $\overset{\alpha}{\lambda}_{\mathrm{iso}}$ in the direction defined by (5) can be calculated[1] (cf. Freund and Ihlemann (2010)):

$$\overset{\alpha}{\lambda}_{\mathrm{iso}} = \sqrt{\overset{\alpha}{\vec{e}} \cdot \hat{\mathbf{C}}\,\overset{\alpha}{\vec{e}}} = \mathrm{III}^{-1/6}\sqrt{\overset{\alpha}{\vec{e}} \cdot \mathbf{C}\,\overset{\alpha}{\vec{e}}} \tag{6}$$

Following Naumann and Ihlemann (2012), the projection of the temperature gradient with regard to the direction vector (5) is also defined:

$$\overset{\alpha}{g}_{\mathrm{R}} = \mathrm{Grad}(\theta(\vec{X}, t)) \cdot \overset{\alpha}{\vec{e}} \tag{7}$$

Now, the angular averaging operator is introduced in which the expression $\sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi = \mathrm{d}\Omega$ denotes the infinitesimal solid angle element[2]:

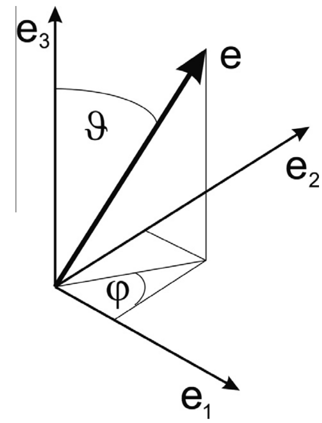

Fig. 1. Unit vector in the orientation space of the reference configuration.

$$x = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \overset{\alpha}{x}(\vartheta, \varphi)\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi \quad \Longleftrightarrow \quad x = A\left[\overset{\alpha}{x}\right] \tag{8}$$

To allow for a compact notation, the double integral is abbreviated as $A[\ldots]$. The argument $\overset{\alpha}{x}$ of this linear operator can be a scalar, a vector or a tensor of any order.

In order to derive the relation between the scalar stress which is linked to the stretch (6) in the direction (5) and an appropriate stress tensor, an additional study is needed.

3. Thermodynamical approach

Thermodynamical consistent constitutive models have to be compatible with the second law of thermodynamics in the form of the Clausius-Duhem inequality[3]:

$$-\rho_{\mathrm{R}}\dot{\psi} + \tilde{\mathbf{T}} \cdot \dot{\mathbf{E}} - \rho_{\mathrm{R}} s\dot{\theta} - \frac{\vec{q}_{\mathrm{R}} \cdot \vec{g}_{\mathrm{R}}}{\theta} \geqslant 0 \tag{9}$$

The scalars ρ_{R}, θ, ψ and s denote the density, the thermodynamic temperature, the specific Helmholtz free energy and the entropy per unit mass related to the reference configuration; $\tilde{\mathbf{T}}$ is the second Piola–Kirchhoff stress tensor and $\dot{\mathbf{E}}$ the material time rate of the Green strain tensor defined in (3); $\vec{g}_{\mathrm{R}} = \mathrm{Grad}(\theta(\vec{X}, t))$ is the temperature gradient and $\vec{q}_{\mathrm{R}}$ the heat flux vector. According to the split of the deformation gradient (1), the free energy, the entropy and the second Piola Kirchhoff stress are assumed to be the sums of two parts – one depending only on volumetric, another only depending on isochoric values:

$$\psi = \psi_{\mathrm{vol}} + \psi_{\mathrm{iso}} \tag{10}$$

$$s = s_{\mathrm{vol}} + s_{\mathrm{iso}} \tag{11}$$

$$\tilde{\mathbf{T}} = \tilde{\mathbf{T}}_{\mathrm{vol}} + \tilde{\mathbf{T}}_{\mathrm{iso}} \tag{12}$$

The volumetric parts are postulated to depend only on the current temperature and the current change in volume but the isochoric parts and the heat flux are represented as follows:

$$\psi_{\mathrm{iso}} = A\left[\overset{\alpha}{\psi}_{\mathrm{iso}}\right] \tag{13}$$

$$s_{\mathrm{iso}} = A\left[\overset{\alpha}{s}_{\mathrm{iso}}\right] \tag{14}$$

[1] The dot is the scalar product. If $\vec{a}, \vec{b}$ are vectors in Cartesian coordinates we have $\vec{a} \cdot \vec{b} = \sum_{k=1}^{3} a_k b_k$.

[2] In their theory, Miehe et al (2004) used the non-linear p-root averaging operator to calculate the non-affine stretches in which p is an additional material parameter: $\langle x \rangle_p = (\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}(\overset{\alpha}{x}(\vartheta, \varphi))^p \sin\vartheta\ \ \mathrm{d}\vartheta\ \ \mathrm{d}\varphi)^{1/p}$.

[3] If $\mathbf{A}, \mathbf{B}$ are tensors of the second order in a Cartesian coordinate system we have $\mathbf{A} \cdot \mathbf{B} = \sum_{i,k=1}^{3} A_{ik} B_{ik}$.

$$\tilde{\mathbf{T}}_{\text{iso}} = A\left[\overset{\alpha}{\tilde{T}}_{\text{iso}}\right] \tag{15}$$

$$\vec{q}_{\text{R}} = A\left[\overset{\alpha}{\vec{q}}_{\text{R}}\right] \tag{16}$$

The variables $\overset{\alpha}{\psi}_{\text{iso}}$, $\overset{\alpha}{s}_{\text{iso}}$, $\overset{\alpha}{\tilde{T}}_{\text{iso}}$ and $\overset{\alpha}{\vec{q}}_{\text{R}}$ denote the specific free energy, the entropy, the second Piola Kirchhoff stress and the heat flux vector belonging to the direction $\overset{\alpha}{\vec{e}}(\vartheta, \varphi)$ (see also Naumann and Ihlemann (2012) in this context). In the most general case of thermoinelastic material behaviour all these directional quantities depend on the histories of deformation and temperature.

Assuming thermoelastic volume behaviour in the form of $\psi_{\text{vol}} = \psi_{\text{vol}}(\text{III}, \theta)$ the material time derivative of (10) leads to the following result:

$$\dot{\psi} = 2\text{III}\frac{\partial \psi_{\text{vol}}}{\partial \text{III}}\mathbf{C}^{-1} \cdot \dot{\mathbf{E}} + \frac{\partial \psi_{\text{vol}}}{\partial \theta}\dot{\theta} + \dot{\psi}_{iso} \tag{17}$$

Inserting (10)–(12) in combination with (17) into the Clausius–Duhem inequality (9) and defining the volumetric parts of the second Piola–Kirchhoff stress and the specific entropy,

$$\tilde{\mathbf{T}}_{\text{vol}} = 2\rho_{\text{R}}\text{III}\frac{\partial \psi_{\text{vol}}}{\partial \text{III}}\mathbf{C}^{-1} \tag{18}$$

$$s_{\text{vol}} = -\frac{\partial \psi_{\text{vol}}}{\partial \theta} \tag{19}$$

a reduced form of the Clausius–Duhem inequality is obtained:

$$-\rho_{\text{R}}\dot{\psi}_{\text{iso}} + \tilde{\mathbf{T}}_{\text{iso}} \cdot \dot{\mathbf{E}} - \rho_{\text{R}} s_{\text{iso}}\dot{\theta} - \frac{\vec{q}_{\text{R}} \cdot \vec{g}_{\text{R}}}{\theta} \geqslant 0 \tag{20}$$

The next task is the reformulation of the isochoric stress power in (20). For physical reasons, the directional stress tensor of the second Piola–Kirchhoff type should satisfy the relation

$$\overset{\alpha}{\tilde{\mathbf{T}}}_{\text{iso}} \cdot \dot{\mathbf{E}} = \overset{\alpha}{\sigma}_{\text{iso}}\frac{\text{d}}{\text{d}t}\overset{\alpha}{\lambda}_{\text{iso}} \tag{21}$$

The scalar $\overset{\alpha}{\sigma}_{\text{iso}}$ is the uniaxial stress related to the isochoric stretch in the direction $\overset{\alpha}{\vec{e}}(\vartheta, \varphi)$. Considering (21) and (6) the material time derivative of the stretch can be computed:

$$\frac{\text{d}}{\text{d}t}\overset{\alpha}{\lambda}_{\text{iso}}(t) = \frac{\text{d}}{\text{d}t}\left(\text{III}(t)^{-1/6}\sqrt{\overset{\alpha}{\vec{e}} \cdot \mathbf{C}(t)\,\overset{\alpha}{\vec{e}}}\right) \tag{22}$$

Application of the product and chain rules of differentiation leads to the intermediate result

$$\frac{\text{d}}{\text{d}t}\overset{\alpha}{\lambda}_{\text{iso}} = -\frac{1}{6}\text{III}^{-1/6}\mathbf{C}^{-1} \cdot \dot{\mathbf{C}}\sqrt{\overset{\alpha}{\vec{e}} \cdot \mathbf{C}\,\overset{\alpha}{\vec{e}}} + \frac{1}{2}\text{III}^{-1/6}\frac{1}{\sqrt{\overset{\alpha}{\vec{e}} \cdot \mathbf{C}\,\overset{\alpha}{\vec{e}}}}\overset{\alpha}{\vec{e}} \cdot \dot{\mathbf{C}}\,\overset{\alpha}{\vec{e}} \tag{23}$$

Rearranging the terms, considering (6) and the formula $\vec{a} \cdot (\mathbf{A}\,\vec{a}) = (\vec{a} \otimes \vec{a}) \cdot \mathbf{A}$ in combination with $\dot{\mathbf{C}} = 2\dot{\mathbf{E}}$ leads to the following relation between the time rates of the Green strain tensor and the isochoric directional stretch:

$$\frac{\text{d}}{\text{d}t}\overset{\alpha}{\lambda}_{\text{iso}} = \left(\frac{1}{\text{III}^{1/3}\overset{\alpha}{\lambda}_{\text{iso}}}\overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}} - \frac{1}{3}\overset{\alpha}{\lambda}_{\text{iso}}\mathbf{C}^{-1}\right) \cdot \dot{\mathbf{E}} \tag{24}$$

Inserting this outcome into (21) motivates the following definition of the directional second Piola–Kirchhoff stress tensor:

$$\overset{\alpha}{\tilde{\mathbf{T}}}_{\text{iso}} = \overset{\alpha}{\sigma}_{\text{iso}}\left(\frac{1}{\text{III}^{1/3}\overset{\alpha}{\lambda}_{\text{iso}}}\overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}} - \frac{1}{3}\overset{\alpha}{\lambda}_{\text{iso}}\mathbf{C}^{-1}\right) \tag{25}$$

A look at relation (25) shows that it is quite close to the expression proposed by Freund and Ihlemann (2009, 2010). When (25) is inserted into (15) to calculate the isochoric part of the second Piola–Kirchhoff stress tensor the following result is obtained:

$$\tilde{\mathbf{T}}_{\text{iso}} = A\left[\overset{\alpha}{\tilde{\text{T}}}_{\text{iso}}\right] = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\overset{\alpha}{\sigma}_{\text{iso}} \times \left(\frac{1}{\text{III}^{1/3}\overset{\alpha}{\lambda}_{\text{iso}}}\overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}} - \frac{1}{3}\overset{\alpha}{\lambda}_{\text{iso}}\mathbf{C}^{-1}\right)\sin\vartheta\,\text{d}\vartheta\,\text{d}\varphi \tag{26}$$

In order to extend the flexibility of the model and to provide better fits to experimental data, Freund and Ihlemann (2009, 2010) introduced an additional weighting factor w in the above equation in the sense of $\tilde{\mathbf{T}}_{\text{iso}} = w/4\pi \ldots$. This factor is different from those which appear in the context of the numerical integration of the averaging operator (cf. Miehe et al. (2004)) as it is neither dependent on the choice of the directions nor on the geometry.

Considering (7) and (16) the scalar product between the temperature gradient and the heat flux vector is reformulated:

$$\begin{aligned}\vec{q}_{\text{R}} \cdot \vec{g}_{\text{R}} &= A\left[\overset{\alpha}{\vec{q}}_{\text{R}}\right] \cdot \vec{g}_{\text{R}} = A\left[\overset{\alpha}{\vec{q}}_{\text{R}} \cdot \vec{g}_{\text{R}}\right] = A\left[(\overset{\alpha}{q}_{\text{R}}\overset{\alpha}{\vec{e}}) \cdot \vec{g}_{\text{R}}\right] \\ &= A\left[\overset{\alpha}{q}_{\text{R}}(\vec{g}_{\text{R}} \cdot \overset{\alpha}{\vec{e}})\right] = A\left[\overset{\alpha}{q}_{\text{R}}\overset{\alpha}{g}_{\text{R}}\right]\end{aligned} \tag{27}$$

If (26) and (27) are inserted into (20) in combination with (13)–(15), (21) and the averaging operator (8), the Clausius–Duhem inequality reads as follows:

$$\frac{1}{4\pi}\int_0^{2\pi} \times \int_0^{\pi}\left(-\rho_{\text{R}}\frac{\text{d}}{\text{d}t}\overset{\alpha}{\psi}_{\text{iso}} + \overset{\alpha}{\sigma}_{\text{iso}}\frac{\text{d}}{\text{d}t}\overset{\alpha}{\lambda}_{\text{iso}} - \rho_{\text{R}}\overset{\alpha}{s}_{\text{iso}}\frac{\text{d}}{\text{d}t}\theta - \frac{1}{\theta}\overset{\alpha}{q}_{\text{R}}\overset{\alpha}{g}_{\text{R}}\right)\sin\vartheta\,\text{d}\vartheta\,\text{d}\varphi \geqslant 0 \tag{28}$$

It has to be non-negative for arbitrary thermomechanical process histories. Since the domain of integration of (28) is fixed and defined by the set union $0 \leqslant \vartheta \leqslant \pi \cup 0 \leqslant \varphi \leqslant 2\pi$, it cannot be concluded that the bracketed integrand has to be non-negative for each direction $\overset{\alpha}{\vec{e}}(\vartheta, \varphi)$ in the orientation space although $\sin\vartheta \geqslant 0$ holds for $0 \leqslant \vartheta \leqslant \pi$. In principle, it is imaginable that the bracketed term is positive for certain directions and negative for others. But, on the other hand, if the one-dimensional material models which are defined by constitutive functions for the variables $\overset{\alpha}{\psi}_{\text{iso}}, \overset{\alpha}{\sigma}_{\text{iso}}, \overset{\alpha}{s}_{\text{iso}}, \overset{\alpha}{q}_{\text{R}}$ in dependence on the histories of $\overset{\alpha}{\lambda}_{\text{iso}}, \theta, \overset{\alpha}{g}_{\text{R}}$ are compatible with their associated Clausius–Duhem inequality

$$-\rho_{\text{R}}\frac{\text{d}}{\text{d}t}\overset{\alpha}{\psi}_{\text{iso}} + \overset{\alpha}{\sigma}_{\text{iso}}\frac{\text{d}}{\text{d}t}\overset{\alpha}{\lambda}_{\text{iso}} - \rho_{\text{R}}\overset{\alpha}{s}_{\text{iso}}\frac{\text{d}}{\text{d}t}\theta - \frac{1}{\theta}\overset{\alpha}{q}_{\text{R}}\overset{\alpha}{g}_{\text{R}} \geqslant 0 \tag{29}$$

then (28) is satisfied and the resulting tensor formulation which is the given by (10)–(16), (18), (19) and (26) is compatible with (20). In the article of Naumann and Ihlemann (2012), the hydrostatic pressure

$$p = -\frac{1}{3}\text{tr}\left(\frac{1}{J}\mathbf{F}\tilde{\mathbf{T}}_{\text{vol}}\mathbf{F}^{\text{T}}\right) = -2\rho_{\text{R}}J\frac{\partial \psi_{\text{vol}}}{\partial \text{III}} \tag{30}$$

which is related to the volumetric part (18) of the stress is also resolved into directional parts. As a result of their model formulation, this leads to a supplementary contribution in the one-dimensional Clausius–Duhem inequality. Such a contribution does not occur in the present approach.

4. Properties of the model

4.1. Structure of the stress tensor

If (26) is transformed to the current configuration and $\mathbf{F}\mathbf{C}^{-1}\mathbf{F}^{\mathrm{T}} = \mathbf{1}$ is taken into account, the Kirchhoff stress tensor $\mathbf{S}_{\mathrm{iso}} = \mathbf{F}\tilde{\mathbf{T}}_{\mathrm{iso}}\mathbf{F}^{\mathrm{T}}$ can be calculated:

$$\mathbf{S}_{\mathrm{iso}} = \frac{1}{4\pi}\int_0^{2\pi} \times \int_0^{\pi} \overset{\alpha}{\sigma}_{\mathrm{iso}} \left(\frac{1}{\mathrm{III}^{1/3}\overset{\alpha}{\lambda}_{\mathrm{iso}}} \mathbf{F}\overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}}\mathbf{F}^{\mathrm{T}} - \frac{1}{3}\overset{\alpha}{\lambda}_{\mathrm{iso}}\mathbf{1} \right) \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi \tag{31}$$

Considering $\mathbf{F}\overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\overset{\alpha}{\vec{e}} \otimes \mathbf{F}\overset{\alpha}{\vec{e}}$ as well as $\mathrm{tr}\left(\mathbf{F}\overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}}\mathbf{F}^{\mathrm{T}}\right) = \mathbf{F}\overset{\alpha}{\vec{e}} \cdot \mathbf{F}\overset{\alpha}{\vec{e}} = \overset{\alpha}{\vec{e}} \cdot \mathbf{F}^{\mathrm{T}}\mathbf{F}\overset{\alpha}{\vec{e}}$ the isochoric directional stretch (6) can be written as follows:

$$\overset{\alpha}{\lambda}_{\mathrm{iso}} = \mathrm{III}^{-1/6}\left(\mathrm{tr}\left(\mathbf{F}\overset{\alpha}{\vec{e}} \otimes \mathbf{F}\overset{\alpha}{\vec{e}}\right)\right)^{1/2} \tag{32}$$

Inserting this expression into (31) leads to the following representation:

$$\mathbf{S}_{\mathrm{iso}} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \times \frac{\overset{\alpha}{\sigma}_{\mathrm{iso}}}{\mathrm{III}^{1/3}\overset{\alpha}{\lambda}_{\mathrm{iso}}} \left(\mathbf{F}\overset{\alpha}{\vec{e}} \otimes \mathbf{F}\overset{\alpha}{\vec{e}} - \frac{1}{3}\mathrm{tr}\left(\mathbf{F}\overset{\alpha}{\vec{e}} \otimes \mathbf{F}\overset{\alpha}{\vec{e}}\right)\mathbf{1} \right) \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi \tag{33}$$

The property[4]

$$\mathrm{tr}\left(\mathbf{F}\overset{\alpha}{\vec{e}} \otimes \mathbf{F}\overset{\alpha}{\vec{e}} - \frac{1}{3}\mathrm{tr}\left(\mathbf{F}\overset{\alpha}{\vec{e}} \otimes \mathbf{F}\overset{\alpha}{\vec{e}}\right)\mathbf{1} \right) = 0 \tag{34}$$

together with (33) leads to $\mathrm{tr}(\mathbf{S}_{\mathrm{iso}}) = 0$ such that the first Piola–Kirchhoff stress $\mathbf{S}_{\mathrm{iso}}$ is purely deviatoric. Computing the tensor $\mathbf{S}_{\mathrm{vol}} = \mathbf{F}\tilde{\mathbf{T}}_{\mathrm{vol}}\mathbf{F}^{\mathrm{T}}$ under consideration of (18), it comes out that it is purely volumetric.

4.2. Structure of the heat flux vector

In order to calculate the resulting heat flux vector, relation (16) must be taken into account in combination with an appropriate one-dimensional heat flow model which is related to the direction $\overset{\alpha}{\vec{e}}(\vartheta, \varphi)$ in the reference configuration:

$$\overset{\alpha}{q}_{\mathrm{R}} = -\kappa(\ldots)\overset{\alpha}{g}_{\mathrm{R}} \tag{35}$$

The scalar $\kappa(\ldots) \geqslant 0$ is the associated heat conductivity which can be constant or variable. A short computation leads to the following relation for the global heat flux vector:

$$\begin{aligned}\vec{q}_{\mathrm{R}} &= A\left[\overset{\alpha}{q}_{\mathrm{R}}\overset{\alpha}{\vec{e}}\right] = -A\left[\kappa(\ldots)\overset{\alpha}{g}_{\mathrm{R}}\overset{\alpha}{\vec{e}}\right] = -A\left[\kappa(\ldots)\left(\vec{g}_{\mathrm{R}} \cdot \overset{\alpha}{\vec{e}}\right)\overset{\alpha}{\vec{e}}\right] \\ &= -A\left[\kappa(\ldots)\overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}}\right]\vec{g}_{\mathrm{R}}\end{aligned} \tag{36}$$

The factor of $\vec{g}_{\mathrm{R}}$ on the right-hand side of (36) is a process-dependent heat conductivity tensor. In the case of $\kappa = \kappa_0 = const$, (36) can be easily evaluated in closed form and leads an isotropic relation between the heat flux vector and the temperature gradient:

$$\begin{aligned}\vec{q}_{\mathrm{R}} &= -\kappa_0 A\left[\overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}}\right]\vec{g}_{\mathrm{R}} \\ &= -\frac{\kappa_0}{3}\left(\vec{e}_1 \otimes \vec{e}_1 + \vec{e}_2 \otimes \vec{e}_2 + \vec{e}_3 \otimes \vec{e}_3\right)\vec{g}_{\mathrm{R}} = -\frac{\kappa_0}{3}\vec{g}_{\mathrm{R}}\end{aligned} \tag{37}$$

The constant $\kappa_0/3$ is the heat conductivity (cf. Naumann and Ihlemann (2012)). If the heat conductivity in (35) depends on the stretch history, (36) is an anisotropic heat conduction law. The anisotropy is process-induced.

4.3. Evaluation for linear thermoelasticity

Under the assumption of small strains, the deformation gradient $\mathbf{F} = \mathbf{1} + \mathbf{H}$ is approximately a unit tensor with $\|\mathbf{H}\| = \sqrt{\mathbf{H} \cdot \mathbf{H}} \ll 1$ and the infinitesimal strain tensor $\mathbf{E}_{\mathrm{lin}}$ reads as follows:

$$\mathbf{E}_{\mathrm{lin}} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}}\right) \tag{38}$$

Using the linearizations $\mathrm{III} \approx 1 + 2\mathrm{tr}(\mathbf{E}_{\mathrm{lin}})$ and $\mathbf{C} \approx \mathbf{1} + 2\mathbf{E}_{\mathrm{lin}}$, it can be shown that the isochoric stretch is determined by the deviator of (38):

$$\overset{\alpha}{\lambda}_{\mathrm{iso}} \approx 1 + \overset{\alpha}{\vec{e}} \cdot \left(\mathbf{E}_{\mathrm{lin}} - \frac{1}{3}\mathrm{tr}(\mathbf{E}_{\mathrm{lin}})\mathbf{1}\right)\overset{\alpha}{\vec{e}} = 1 + \overset{\alpha}{\vec{e}} \cdot \mathbf{E}_{\mathrm{lin}}^{\mathrm{D}}\overset{\alpha}{\vec{e}} \tag{39}$$

If the stress $\overset{\alpha}{\sigma}_{\mathrm{iso}}$ depends linearly on the strain $\overset{\alpha}{\varepsilon}_{\mathrm{iso}} = \overset{\alpha}{\lambda}_{\mathrm{iso}} - 1$ and $\mathrm{tr}(\overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}}) = 1$ is considered, the linearization of (33) reads as

$$\mathbf{S}_{\mathrm{iso}} \approx \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \overset{\alpha}{\sigma}_{\mathrm{iso}}\left(\overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}} - \frac{1}{3}\mathbf{1}\right)\sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi \tag{40}$$

Applying the transformation $\mathbf{S}_{\mathrm{vol}} = \mathbf{F}\tilde{\mathbf{T}}_{\mathrm{vol}}\mathbf{F}^{\mathrm{T}}$ to (18) and assuming small volume changes such that the quadratic approximation

$$\psi_{\mathrm{vol}}(\mathrm{III},\theta) = \psi_{\mathrm{vol}}(1,\theta) + \frac{\psi''_{\mathrm{vol}}(1,\theta)}{2}(\mathrm{III}-1)^2 + O\left((\mathrm{III}-1)^3\right) \tag{41}$$

is applicable,[5] the linearization of the hydrostatic stress can be easily calculated. In order to guarantee that the undeformed configuration is free of stress $\psi'_{\mathrm{vol}}(1,\theta) = 0$ is assumed in (41). Defining the temperature-dependent modulus $K(\theta) = 4\rho_{\mathrm{R}}\psi''_{\mathrm{vol}}(1,\theta)$, taking $\mathrm{III} \approx 1 + 2\mathrm{tr}(\mathbf{E}_{\mathrm{lin}})$ into account and omitting higher order terms, with (18) the following result is obtained:

$$\mathbf{S}_{\mathrm{vol}} = K(\theta)\ \mathrm{tr}(\mathbf{E}_{\mathrm{lin}})\mathbf{1} \tag{42}$$

In the case of a linear elastic material model for the direction $\overset{\alpha}{\vec{e}}(\vartheta, \varphi)$ the related free energy

$$\rho_{\mathrm{R}}\overset{\alpha}{\psi}_{\mathrm{iso}} = \mu(\theta)\left(\overset{\alpha}{\lambda}_{\mathrm{iso}} - 1\right)^2 \tag{43}$$

in combination with the isothermal form of (29) leads to the relation for the stress:

$$\overset{\alpha}{\sigma}_{\mathrm{iso}} = \rho_{\mathrm{R}}\frac{\partial\overset{\alpha}{\psi}_{\mathrm{iso}}}{\partial\overset{\alpha}{\lambda}_{\mathrm{iso}}} = 2\mu(\theta)\left(\overset{\alpha}{\lambda}_{\mathrm{iso}} - 1\right) \tag{44}$$

The function $\mu(\theta) > 0$ is a temperature-dependent elasticity parameter. Taking all results into account, the total stress can be written as follows:

$$\mathbf{S} = K(\theta)\ \mathrm{tr}(\mathbf{E}_{\mathrm{lin}})\mathbf{1} + \frac{\mu(\theta)}{2\pi}\int_0^{2\pi} \times \int_0^{\pi}\left(\overset{\alpha}{\lambda}_{\mathrm{iso}} - 1\right)\left(\overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}} - \frac{1}{3}\mathbf{1}\right)\sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi \tag{45}$$

[4] The trace of a second order tensor in an orthogonal Cartesian coordinate system is $\mathrm{tr}(\mathbf{B}) = B_{11} + B_{22} + B_{33}$.

[5] The Landau symbol is defined as follows: $f(x) = O(x^2) \iff |f(x)| < Cx^2$ for $x \to 0$.

Since $\mathbf{T} = (1/J)\mathbf{S} \approx (1 - \mathrm{tr}(\mathbf{E}_{\mathrm{lin}}))\mathbf{S} \approx \mathbf{S}$ holds in the case of small strains, the stress $\mathbf{S}$ in (45) is approximately equal to the Cauchy stress tensor $\mathbf{T}$.

In order to compute the responses of (45) for hydrostatic compression, shear and uniaxial tension, both the stress and the linearized strain tensor have to be specified and the integral be evaluated under consideration of (5). To this end, the following dyadic product is needed:

$$\begin{aligned}\overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}} = {} & \sin^2\vartheta\cos^2\varphi\,\vec{e}_1 \otimes \vec{e}_1 \\ & + \sin^2\vartheta\sin\varphi\cos\varphi\left(\vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1\right) \\ & + \sin^2\vartheta\sin^2\varphi\,\vec{e}_2 \otimes \vec{e}_2 \\ & + \sin\vartheta\cos\vartheta\cos\varphi\left(\vec{e}_1 \otimes \vec{e}_3 + \vec{e}_3 \otimes \vec{e}_1\right) \\ & + \sin\vartheta\cos\vartheta\sin\varphi\left(\vec{e}_2 \otimes \vec{e}_3 + \vec{e}_3 \otimes \vec{e}_2\right) + \cos^2\vartheta\,\vec{e}_3 \\ & \otimes \vec{e}_3 \end{aligned} \tag{46}$$

4.3.1. Hydrostatic compression

In this case, $\varepsilon_{\mathrm{vol}}$ denotes the volume strain and p the pressure. The related strain and stress tensors read as

$$\mathbf{E}_{\mathrm{lin}} = \frac{\varepsilon_{\mathrm{vol}}}{3}\left(\vec{e}_1 \otimes \vec{e}_1 + \vec{e}_2 \otimes \vec{e}_2 + \vec{e}_3 \otimes \vec{e}_3\right) \tag{47}$$

$$\mathbf{S} = -p\left(\vec{e}_1 \otimes \vec{e}_1 + \vec{e}_2 \otimes \vec{e}_2 + \vec{e}_3 \otimes \vec{e}_3\right) \tag{48}$$

Considering (39), (45)–(48) as well as $\mathbf{E}_{\mathrm{lin}}^{\mathrm{D}} = \mathbf{0}$, the results are as follows:

$$\overset{\alpha}{\lambda}_{\mathrm{iso}} - 1 = 0 \tag{49}$$

$$-p = K(\theta)\varepsilon_{\mathrm{vol}} \tag{50}$$

This outcome shows that the function $K(\theta)$ is the compression modulus.

4.3.2. Simple shear

Under shear loading, γ and τ denote the shear angle and shear stress such that the related tensors are given by

$$\mathbf{E}_{\mathrm{lin}} = \frac{\gamma}{2}\left(\vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1\right) \tag{51}$$

$$\mathbf{S} = \tau\left(\vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1\right) \tag{52}$$

Considering (39), (45), (46), (51), and (52) the final result reads as

$$\overset{\alpha}{\lambda}_{\mathrm{iso}} - 1 = \gamma\sin^2\vartheta\sin\varphi\cos\varphi \tag{53}$$

$$\tau = \frac{2\mu(\theta)}{15}\gamma \tag{54}$$

The last expression shows that the factor $G(\theta) = 2\mu(\theta)/15$ is the shear modulus.

4.3.3. Uniaxial tension

The axial strain is denoted as ε, the lateral strain as $\varepsilon_{\mathrm{lat}}$, the axial stress as σ and the related tensors are written as

$$\mathbf{E}_{\mathrm{lin}} = \varepsilon\vec{e}_1 \otimes \vec{e}_1 + \varepsilon_{\mathrm{lat}}\left(\vec{e}_2 \otimes \vec{e}_2 + \vec{e}_3 \otimes \vec{e}_3\right) \tag{55}$$

$$\mathbf{S} = \sigma\vec{e}_1 \otimes \vec{e}_1 \tag{56}$$

Consideration of relations (39), (45), (46), (55), and (56) leads to the final results:

$$\overset{\alpha}{\lambda}_{\mathrm{iso}} - 1 = (\varepsilon - \varepsilon_{\mathrm{lat}})\left(\sin^2\vartheta\cos^2\varphi - \frac{1}{3}\right) \tag{57}$$

$$\varepsilon_{\mathrm{lat}} = -\frac{45K - 4\mu}{90K + 4\mu}\varepsilon \tag{58}$$

$$\sigma = \mu\frac{18K}{45K + 2\mu}\varepsilon \tag{59}$$

Taking a look at (58) and (59) it is apparent that $\nu = (45K - 4\mu)/(90K + 4\mu)$ is the Poisson ratio and $E = 18\mu K/(45K + 2\mu)$ the Young modulus. It can be easily verified that $\mu > 0$ and $K > 0$ imply $E > 0$, $G > 0$ and $-1 < \nu < 1/2$. If the volumetric/isochoric decomposition of the deformation gradient and the free energy is not introduced in the model formulation then the Poisson ratio would have a constant value of $1/4$ (e.g. Ostwald et al. (2010)).

In the case of nearly incompressible material behaviour, $K \gg \mu$ applies. Then, the evaluation of (58) and (59) leads to $\varepsilon_{\mathrm{lat}} = -\varepsilon/2$ and to

$$\sigma = \frac{2}{5}\mu(\theta)\varepsilon \tag{60}$$

Interpreting $E = 2/5\mu$ as Young modulus of the incompressible material and comparing this value with the modulus $\overset{\alpha}{E} = 2\mu$ in (44), the global stress-strain response is 5 times weaker than the response which was taken for one single direction. This interesting outcome was also derived by Freund and Ihlemann (2010).

5. Evaluation for nonlinear elasticity of the Mooney–Rivlin type

As pointed out, the directional approach can be applied to transfer uniaxial material models to three spatial dimensions. Owing to the inherent nonlinearities of this approach, the averaging operator (8) is usually approximated by discrete sums. Different discretisation techniques as well as their implications are analysed by Ehret et al. (2010). The reader is also referred to Bazant and Oh (1986), Miehe et al. (2004) or Freund and Ihlemann (2010). In order to determine the accuracy of such approximations or to understand the functional principle of the averaging operator in more detail, exact solutions for relevant one-dimensional material models and load situations are required.

To analyse the response behaviour of the large strain formulation (31) for the special case of nonlinear elasticity, the constraint of incompressibility is introduced. It corresponds to $J = 1$ and implies the following relation for the Cauchy stress tensor:

$$\begin{aligned}\mathbf{T} &= -p\mathbf{1} + \mathbf{S}_{\mathrm{iso}} \\ &= -p\mathbf{1} + \frac{1}{4\pi}\int_0^{2\pi} \\ &\quad\times \int_0^{\pi} \overset{\alpha}{\sigma}_{\mathrm{iso}}\left(\frac{1}{\overset{\alpha}{\lambda}_{\mathrm{iso}}}\mathbf{F}\overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}}\mathbf{F}^{\mathrm{T}} - \frac{1}{3}\overset{\alpha}{\lambda}_{\mathrm{iso}}\mathbf{1}\right)\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi \end{aligned} \tag{61}$$

The factor p is the constitutively undetermined pressure. In order to enable the evaluation of the double integral in closed form, the one-dimensional tension/compression relation of the usual Mooney–Rivlin model is applied for the directions $\overset{\alpha}{\vec{e}}(\vartheta, \varphi)$. This leads to the following expressions for the free energy and the stress; $\mu_1, \mu_2 \geqslant 0$ are material constants:

$$\rho_{\mathrm{R}}\overset{\alpha}{\psi}_{\mathrm{iso}} = \mu_1\left(\overset{\alpha}{\lambda}{}^2_{\mathrm{iso}} + \frac{2}{\overset{\alpha}{\lambda}_{\mathrm{iso}}} - 3\right) + \mu_2\left(\frac{1}{\overset{\alpha}{\lambda}{}^2_{\mathrm{iso}}} + 2\overset{\alpha}{\lambda}_{\mathrm{iso}} - 3\right) \tag{62}$$

$$\overset{\alpha}{\sigma}_{\text{iso}} = \rho_{\text{R}} \frac{\partial \overset{\alpha}{\psi}_{\text{iso}}}{\partial \overset{\alpha}{\lambda}_{\text{iso}}} = 2\mu_1 \left(\overset{\alpha}{\lambda}_{\text{iso}} - \overset{\alpha}{\lambda}{}_{\text{iso}}^{-2} \right) + 2\mu_2 \left(1 - \overset{\alpha}{\lambda}{}_{\text{iso}}^{-3} \right) \tag{63}$$

The model defined by (61) and (63) will be evaluated for uniaxial tension/compression in the global $\vec{e}_3$-direction as well as for equibiaxial tension in the global $\vec{e}_1/\vec{e}_2$-plane. It should be mentioned that Carol et al. (2004) postulated different equations on the directional level in order to obtain a macroscopic stress-strain response of the Mooney–Rivlin type. Interested readers are referred to this comprehensive article.

5.1. Uniaxial tension/compression

In this case, the deformation gradient, the Cauchy stress and the deviatoric isochoric stress tensor are represented as follows:

$$\mathbf{F} = \lambda^{-1/2} \left(\vec{e}_1 \otimes \vec{e}_1 + \vec{e}_2 \otimes \vec{e}_2 \right) + \lambda \vec{e}_3 \otimes \vec{e}_3 \tag{64}$$

$$\mathbf{T} = \sigma \vec{e}_3 \otimes \vec{e}_3 \tag{65}$$

$$\mathbf{S}_{\text{iso}} = s \left(-\frac{1}{2} \left(\vec{e}_1 \otimes \vec{e}_1 + \vec{e}_2 \otimes \vec{e}_2 \right) + \vec{e}_3 \otimes \vec{e}_3 \right) \tag{66}$$

In order to calculate the stress σ, the relations (61), (65), and (66) suggest $0 = -p - s/2$ and $\sigma = -p + s$. This leads to

$$\sigma = \frac{3}{2} s \tag{67}$$

Taking (5) and (64) into account, the following relations are obtained:

$$\mathbf{F} \overset{\alpha}{\vec{e}} = \lambda^{-1/2} \sin\vartheta \left(\cos\varphi \ \vec{e}_1 + \sin\varphi \ \vec{e}_2 \right) + \lambda \cos\vartheta \ \vec{e}_3 \tag{68}$$

$$\mathbf{F} \overset{\alpha}{\vec{e}} \cdot \mathbf{F} \overset{\alpha}{\vec{e}} = \lambda^{-1} \sin^2\vartheta + \lambda^2 \cos^2\vartheta \tag{69}$$

$$\overset{\alpha}{\lambda}_{\text{iso}} = \sqrt{\lambda^{-1} \sin^2\vartheta + \lambda^2 \cos^2\vartheta} \tag{70}$$

Considering (61), (63), (67), and (68) as well as $s = \vec{e}_3 \cdot \mathbf{S}_{\text{iso}} \vec{e}_3$, $\vec{e}_3 \cdot (\mathbf{F} \overset{\alpha}{\vec{e}} \otimes \overset{\alpha}{\vec{e}} \mathbf{F}^{\text{T}}) \vec{e}_3 = \lambda^2 \cos^2\vartheta$ and $\vec{e}_3 \cdot \mathbf{1} \vec{e}_3 = 1$, the scalar Cauchy stress can be straightforwardly computed:

$$\sigma = \frac{1}{2} A \left[\frac{\overset{\alpha}{\sigma}_{\text{iso}}}{\overset{\alpha}{\lambda}_{\text{iso}}} \left(2\lambda^2 \cos^2\vartheta - \frac{\sin^2\vartheta}{\lambda} \right) \right] \tag{71}$$

Since the integrand of (71) is independent on the angle φ, the related integration leads to a factor of 2π. Then, (71) is equal to

$$\begin{aligned} \sigma = {} & \frac{\mu_1}{2} \int_0^\pi \left(1 - \left(\lambda^2 \cos^2\vartheta + \frac{\sin^2\vartheta}{\lambda} \right)^{-3/2} \right) \\ & \times \left(2\lambda^2 \cos^2\vartheta - \frac{\sin^2\vartheta}{\lambda} \right) \sin\vartheta \, d\vartheta \\ & + \frac{\mu_2}{2} \int_0^\pi \left(\left(\lambda^2 \cos^2\vartheta + \frac{\sin^2\vartheta}{\lambda} \right)^{-1/2} \right. \\ & \left. - \left(\lambda^2 \cos^2\vartheta + \frac{\sin^2\vartheta}{\lambda} \right)^{-2} \right) \left(2\lambda^2 \cos^2\vartheta - \frac{\sin^2\vartheta}{\lambda} \right) \sin\vartheta \, d\vartheta \end{aligned}$$

or to

$$\begin{aligned} \sigma = {} & \frac{\mu_1}{2} \int_{-1}^{1} \left(1 - \left((\lambda^2 - \lambda^{-1}) x^2 + \lambda^{-1} \right)^{-3/2} \right) \left((2\lambda^2 + \lambda^{-1}) x^2 - \lambda^{-1} \right) dx \\ & + \frac{\mu_2}{2} \int_{-1}^{1} \left(\left((\lambda^2 - \lambda^{-1}) x^2 + \lambda^{-1} \right)^{-1/2} \right. \\ & \left. - \left((\lambda^2 - \lambda^{-1}) x^2 + \lambda^{-1} \right)^{-2} \right) \left((2\lambda^2 + \lambda^{-1}) x^2 - \lambda^{-1} \right) \, dx \end{aligned} \tag{72}$$

when the substitution $x = -\cos\vartheta$ is applied. In order to evaluate (72) in closed form, three cases must be distinguished:

No deformation $\quad \lambda^2 - \lambda^{-1} = 0 \iff \lambda = 1$

Tension : $\quad \lambda^2 - \lambda^{-1} > 0 \iff \lambda > 1$

Compression $\quad \lambda^2 - \lambda^{-1} < 0 \iff \lambda < 1$

In the most simple case of $\lambda = 1$, (72) leads to $\sigma = 0$. In the case of tension, the integral can be solved when $y = x\sqrt{\lambda^2 - \lambda^{-1}}$ is substituted. Under compression, $y = x\sqrt{\lambda^{-1} - \lambda^2}$ must be applied. The solutions of the related integrals can be found in the formulary of Bronstein and Semendjajew (1984). The final outcome of these calculations reads as follows:

$$\sigma_{\text{R}} = \begin{cases} \mu_1 \left(\frac{2}{3}(\lambda - \lambda^{-2}) + \frac{3}{\lambda^2 - \lambda^{-1}} - \frac{2\lambda + \lambda^{-2}}{(\lambda^2 - \lambda^{-1})^{3/2}} \operatorname{arsinh}\left(\sqrt{\lambda^3 - 1} \right) \right) + \mu_2 \left(\frac{\lambda^2 + 2\lambda^{-1}}{\lambda^2 - \lambda^{-1}} \right. & \\ \left. \quad - \frac{4 - \lambda^{-3}}{2(\lambda^2 - \lambda^{-1})^{3/2}} \operatorname{arsinh}(\sqrt{\lambda^3 - 1}) - \frac{\lambda^{1/2}(\lambda + 2\lambda^{-2})}{2(\lambda^2 - \lambda^{-1})^{3/2}} \arctan(\sqrt{\lambda^3 - 1}) \right) & \text{if } \lambda > 1 \\ 0 & \text{if } \lambda = 1 \\ \mu_1 \left(\frac{2}{3}(\lambda - \lambda^{-2}) + \frac{3}{\lambda^2 - \lambda^{-1}} + \frac{2\lambda + \lambda^{-2}}{(\lambda^{-1} - \lambda^2)^{3/2}} \arcsin(\sqrt{1 - \lambda^3}) \right) + \mu_2 \left(\frac{\lambda^2 + 2\lambda^{-1}}{\lambda^2 - \lambda^{-1}} \right. & \\ \left. \quad + \frac{4 - \lambda^{-3}}{2(\lambda^{-1} - \lambda^2)^{3/2}} \arcsin(\sqrt{1 - \lambda^3}) + \frac{\lambda^{1/2}(\lambda + 2\lambda^{-2})}{2(\lambda^{-1} - \lambda^2)^{3/2}} \operatorname{artanh}(\sqrt{1 - \lambda^3}) \right) & \text{if } \lambda < 1 \end{cases} \tag{73}$$

The variable $\sigma_{\text{R}} = \sigma/\lambda$ is the related component of the first Piola–Kirchhoff stress tensor:

$$\mathbf{T}_{\text{R}} = J \mathbf{T} \mathbf{F}^{\text{T}-1} = \frac{\sigma}{\lambda} \vec{e}_3 \otimes \vec{e}_3 = \sigma_{\text{R}} \vec{e}_3 \otimes \vec{e}_3 \tag{74}$$

Comparing (73) with the related stress-stretch relation of a single direction as defined in (63) completely different mathematical forms are observed when $\overset{\alpha}{\lambda}_{\text{iso}}$ is replaced by λ:

$$\sigma_{\text{R}} = 2\mu_1 (\lambda - \lambda^{-2}) + 2\mu_2 (1 - \lambda^{-3})$$

The associated stretch–stress curves are plotted in Figs. 2 and 3 for different values of the material parameters. The stress response of the directional model is much weaker than that of the original one-dimensional model. This behaviour is representative for the affine approach but can be changed in a consistent manner when a non-affine model is used as proposed, for example, by Miehe et al. (2004).

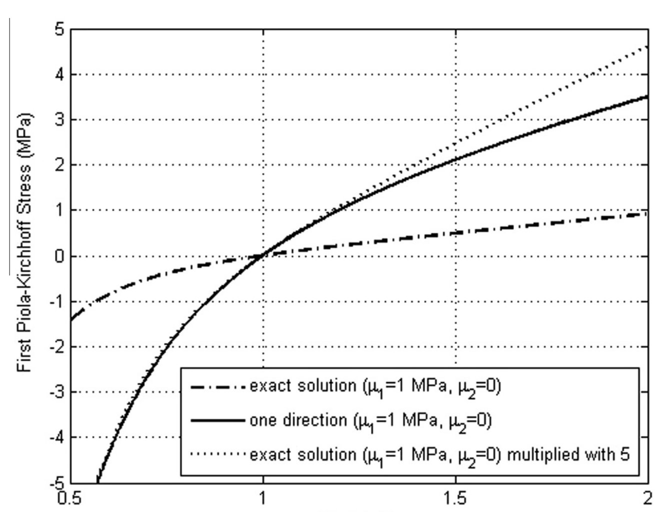

Fig. 2. Stress–stretch responses for $\mu_1 = 1$ MPa, $\mu_2 = 0$ under uniaxial tension/compression.

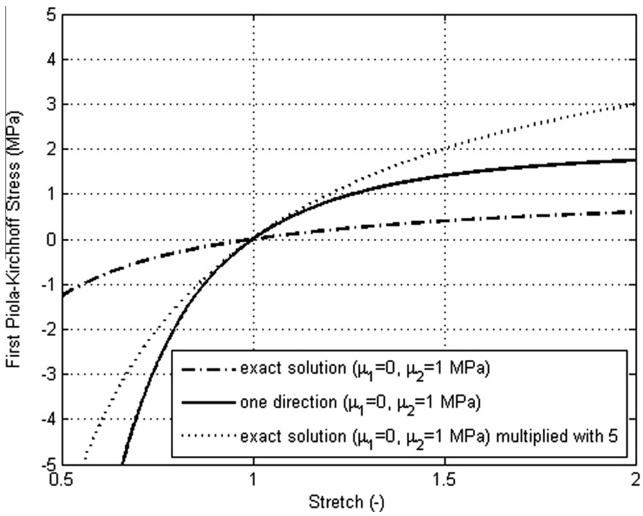

Fig. 3. Stress–stretch responses for $\mu_1 = 0$, $\mu_2 = 1$ MPa under uniaxial tension/compression.

For comparison, the exact solution (73) has also been multiplied with a factor of 5 and presented as dotted curve in the figures. The origin of this number is explained in the text below the linear stress strain relation (60). Freund and Ihlemann (2010) used this value for the parameter w (cf. Freund and Ihlemann (2010) and the discussion in Section 3 of this paper) to guarantee that the stresses of the directional model are comparable with those resulting from the underlying one-dimensional model. In the vicinity of $\lambda = 1$, the multiplied stress response (dotted curve) approximates that of the one-dimensional model. It should be remarked that the theory which is applied in the present paper contains no room for such a parameter. It should also be mentioned, that the parameter w has a completely different meaning than the weighting factors w^i which are introduced for the numerical integration of the angular averaging operator (e.g. Miehe et al. (2004)).

5.2. Equibiaxial tension

In this situation, the tensors $\mathbf{F}$, $\mathbf{T}$ and $\mathbf{S}_{iso}$ are expressed as follows:

$$\mathbf{F} = \lambda\left(\vec{e}_1 \otimes \vec{e}_1 + \vec{e}_2 \otimes \vec{e}_2\right) + \lambda^{-2}\vec{e}_3 \otimes \vec{e}_3 \tag{75}$$

$$\mathbf{T} = \sigma\left(\vec{e}_1 \otimes \vec{e}_1 + \vec{e}_2 \otimes \vec{e}_2\right) \tag{76}$$

$$\mathbf{S}_{iso} = s_1\vec{e}_1 \otimes \vec{e}_1 + s_2\vec{e}_2 \otimes \vec{e}_2 + s_3\vec{e}_3 \otimes \vec{e}_3 \tag{77}$$

Using the representation (5) of the vector $\overset{\alpha}{\vec{e}}(\vartheta, \varphi)$ the listed relations are obtained:

$$\mathbf{F}\overset{\alpha}{\vec{e}} = \lambda \sin\vartheta(\cos\varphi\ \vec{e}_1 + \sin\varphi\ \vec{e}_2) + \lambda^{-2}\cos\vartheta\ \vec{e}_3 \tag{78}$$

$$\mathbf{F}\overset{\alpha}{\vec{e}} \cdot \mathbf{F}\overset{\alpha}{\vec{e}} = \lambda^2\sin^2\vartheta + \lambda^{-4}\cos^2\vartheta \tag{79}$$

$$\overset{\alpha}{\lambda}_{iso} = \sqrt{\lambda^2\sin^2\vartheta + \lambda^{-4}\cos^2\vartheta} \tag{80}$$

$$\begin{aligned}\mathbf{F}\overset{\alpha}{\vec{e}} \otimes \mathbf{F}\overset{\alpha}{\vec{e}} = {} & \lambda^2\sin^2\vartheta\cos^2\varphi\vec{e}_1 \otimes \vec{e}_1 \\ & + \lambda^2\sin^2\vartheta\sin\varphi\cos\varphi\left(\vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1\right) \\ & + \lambda^{-1}\sin\vartheta\cos\vartheta\cos\varphi\left(\vec{e}_1 \otimes \vec{e}_3 + \vec{e}_3 \otimes \vec{e}_1\right) \\ & + \lambda^2\sin^2\vartheta\sin^2\varphi\vec{e}_2 \otimes \vec{e}_2 \\ & + \lambda^{-1}\sin\vartheta\cos\vartheta\sin\varphi\left(\vec{e}_2 \otimes \vec{e}_3 + \vec{e}_3 \otimes \vec{e}_2\right) \\ & + \lambda^{-4}\cos^2\vartheta\vec{e}_3 \otimes \vec{e}_3\end{aligned} \tag{81}$$

Inserting (77), (79), and (81) into (61) the components of the tensor $\mathbf{S}_{iso}$ can be computed:

$$\begin{aligned}s_1 &= A\left[\frac{\overset{\alpha}{\sigma}_{iso}}{\overset{\alpha}{\lambda}_{iso}}\left(\lambda^2\sin^2\vartheta\left(\cos^2\varphi - \frac{1}{3}\right) - \frac{1}{3}\lambda^{-4}\cos^2\vartheta\right)\right] \\ s_2 &= A\left[\frac{\overset{\alpha}{\sigma}_{iso}}{\overset{\alpha}{\lambda}_{iso}}(\lambda^2\sin^2\vartheta\left(\sin^2\varphi - \frac{1}{3}\right) - \frac{1}{3}\lambda^{-4}\cos^2\vartheta)\right] \\ s_3 &= A\left[\frac{\overset{\alpha}{\sigma}_{iso}}{\overset{\alpha}{\lambda}_{iso}}(\frac{2}{3}\lambda^{-4}\cos^2\vartheta - \frac{1}{3}\lambda^2\sin^2\vartheta)\right]\end{aligned} \tag{82}$$

Since the shear components of (81) depend linearly on $\sin\varphi$, $\cos\varphi$ or $\sin\varphi\cos\varphi$ and the factors $\overset{\alpha}{\sigma}_{iso}/\overset{\alpha}{\lambda}_{iso}$ do not depend on the angle φ, the following statements are valid:

$$\begin{aligned}A\left[\frac{\overset{\alpha}{\sigma}_{iso}}{\overset{\alpha}{\lambda}_{iso}}\left(\lambda^2\sin^2\vartheta\sin\varphi\cos\varphi\right)\right] &= A\left[\frac{\overset{\alpha}{\sigma}_{iso}}{\overset{\alpha}{\lambda}_{iso}}\left(\frac{\sin\vartheta\cos\vartheta\cos\varphi}{\lambda}\right)\right] \\ &= A\left[\frac{\overset{\alpha}{\sigma}_{iso}}{\overset{\alpha}{\lambda}_{iso}}\left(\frac{\sin\vartheta\cos\vartheta\sin\varphi}{\lambda}\right)\right] = 0\end{aligned}$$

Taking a look at (82) the integration with regard to the angle φ can be easily carried out and leads to $s_1 = s_2 = -s_3/2 =: s$ with

$$s = \frac{1}{6}\int_0^\pi \frac{\overset{\alpha}{\sigma}_{iso}}{\overset{\alpha}{\lambda}_{iso}}\left(\frac{\lambda^2}{2}\sin^2\vartheta - \lambda^{-4}\cos^2\vartheta\right)\sin\vartheta d\vartheta \tag{83}$$

Inserting this result into (61) and consideration of (76) and (77), the component σ of the Cauchy stress can be computed: $\sigma = -p + s$ and $0 = -p - 2s$ lead to $\sigma = 3s$ or with (83) to

$$\sigma = \frac{1}{2}\int_0^\pi \frac{\overset{\alpha}{\sigma}_{iso}}{\overset{\alpha}{\lambda}_{iso}}\left(\frac{\lambda^2}{2}\sin^2\vartheta - \lambda^{-4}\cos^2\vartheta\right)\sin\vartheta d\vartheta \tag{84}$$

Taking the underlying one-dimensional model (63) into account and substituting $x = -\cos\vartheta$ in (84) a short calculation leads to:

$$\begin{aligned}\sigma = {} & \frac{\mu_1}{2}\int_{-1}^{1}\left(1 - (\lambda^2 - (\lambda^2 - \lambda^{-4})x^2)^{-3/2}\right)(\lambda^2 - (\lambda^2 + 2\lambda^{-4})x^2)dx \\ & + \frac{\mu_2}{2}\int_{-1}^{1}\left((\lambda^2 - (\lambda^2 - \lambda^{-4})x^2)^{-1/2}\right. \\ & \left. - (\lambda^2 - (\lambda^2 - \lambda^{-4})x^2)^{-2}\right)(\lambda^2 - (\lambda^2 + 2\lambda^{-4})x^2)dx\end{aligned} \tag{85}$$

To carry out the integration of (85) in closed form, the condition $\lambda^2 - \lambda^{-4} > 0$ which holds for equibiaxial tension has to be considered and the substitution $y = x\sqrt{\lambda^2 - \lambda^{-4}}$ be applied. All resulting integrals can be found in the formulary of Bronstein and Semendjajew (1984) such that the stress response $\mathbf{T}_R = J\mathbf{T}\mathbf{F}^{T-1} = \sigma/\lambda(\vec{e}_1 \otimes \vec{e}_1 + \vec{e}_2 \otimes \vec{e}_2)$ can be calculated:

$$\sigma_R = \frac{\sigma}{\lambda} = \frac{\sigma_1 + \sigma_2}{\lambda} \tag{86}$$

$$\sigma_1 = \mu_1\left(\frac{\lambda^4+2\lambda^{-2}}{\lambda^2-\lambda^{-4}} - \frac{\lambda^2+2\lambda^{-4}}{3} - \frac{\lambda^2+2\lambda^{-4}}{(\lambda^2-\lambda^{-4})^{3/2}}\arcsin(\sqrt{1-\lambda^{-6}})\right)$$
$$\sigma_2 = \mu_2\left(\frac{2+\lambda^{-6}}{\lambda^2-\lambda^{-4}} + \frac{\lambda^4-4\lambda^{-2}}{2(\lambda^2-\lambda^{-4})^{3/2}}\arcsin(\sqrt{1-\lambda^{-6}}) - \frac{2\lambda+\lambda^{-5}}{2(\lambda^2-\lambda^{-4})^{3/2}}\operatorname{artanh}(\sqrt{1-\lambda^{-6}})\right) \tag{87}$$

The response behaviour of the component of the first Piola Kirchhoff stress tensor as function of the stretch is plotted for different values of the material constants in Figs. 4 and 5. For comparison, the equibiaxial response of the Mooney–Rivlin model in also presented in the figures. This relation can be found in standard textbooks of rubber elasticity and reads as

$$\sigma_R = 2\mu_1(\lambda - \lambda^{-5}) + 2\mu_2(\lambda^3 - \lambda^{-3}) \tag{88}$$

Taking a look at the figures, the stress responses which result from the directional approach are much weaker than those of the Mooney–Rivlin model under equibiaxial tension (88). If the exact solutions of the directional model are multiplied with a factor of 5 as discussed in the last section, the Mooney–Rivlin responses are approximated in the vicinity of $\lambda = 1$. The most interesting result of this investigation is the fact that the curvature of the equibiaxial response in the case of $\mu_1 = 0, \mu_2 = 1$ MPa is totally different from that of the Mooney–Rivlin model.

5.3. Remark

In these considerations, the directional model is hyperelastic, incompressible and isotropic and its strain energy per unit mass is given by (62). Setting $x = \cos\vartheta$ and comparing the directional stretches under uniaxial tension/compression (70) with those under equibiaxial tension (80) the following relations are obtained:

$$\overset{\alpha}{\lambda}_{\text{iso,uni}}(\lambda) = \sqrt{\lambda^{-1}\sin^2\vartheta + \lambda^2\cos^2\vartheta} = \lambda^{-1/2}\sqrt{1+(\lambda^3-1)x^2} \tag{89}$$

$$\overset{\alpha}{\lambda}_{\text{iso,eb}}(\lambda) = \sqrt{\lambda^2\sin^2\vartheta + \lambda^{-4}\cos^2\vartheta} = \lambda\sqrt{1+(\lambda^{-6}-1)x^2} \tag{90}$$

Comparing (89) with (90) leads to $\overset{\alpha}{\lambda}_{\text{iso,uni}}(\lambda^{-2}) = \overset{\alpha}{\lambda}_{\text{iso,eb}}(\lambda)$. Because of this equivalence, the strain energy densities under uniaxial tension/compression and under equibiaxial tension are connected by the relation $\overset{\alpha}{\psi}_{\text{iso,uni}}(\lambda^{-2}) = \overset{\alpha}{\psi}_{\text{iso,eb}}(\lambda)$. It can be computed by inserting (89) and (90) into (62). The application of the angular averaging operator (8) to this result then leads to $\psi_{\text{iso,uni}}(\lambda^{-2}) = \psi_{\text{iso,eb}}(\lambda)$. The principle of conservation of energy which is applicable in the case of elasticity demands the equality between the increment of work of the external forces ($\sigma_{R,\text{uni}}(\lambda)\mathrm{d}\lambda$ under uniaxial and $2\sigma_{R,\text{eb}}(\lambda)\mathrm{d}\lambda$ under equibiaxial loadings) and the change in the strain energy per unit volume $\rho_R \mathrm{d}\psi_{\text{iso,uni}}(\lambda)$ or $\rho_R \mathrm{d}\psi_{\text{iso,eb}}(\lambda)$. This leads to

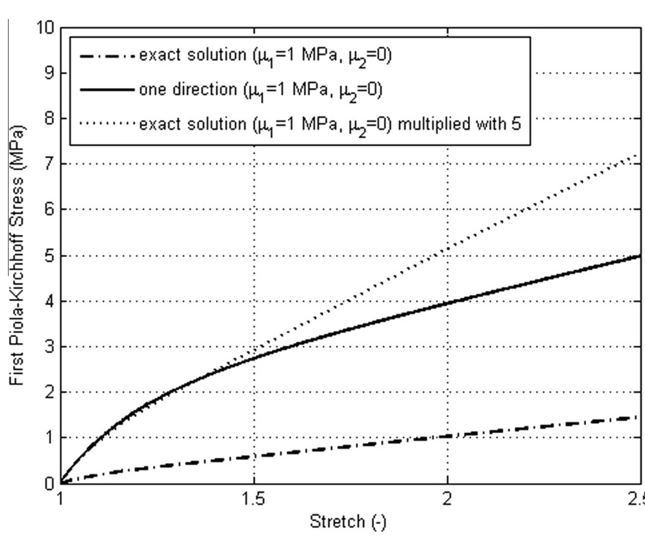

Fig. 4. Stress–stretch responses for $\mu_1 = 1$ MPa, $\mu_2 = 0$ under equibiaxial tension.

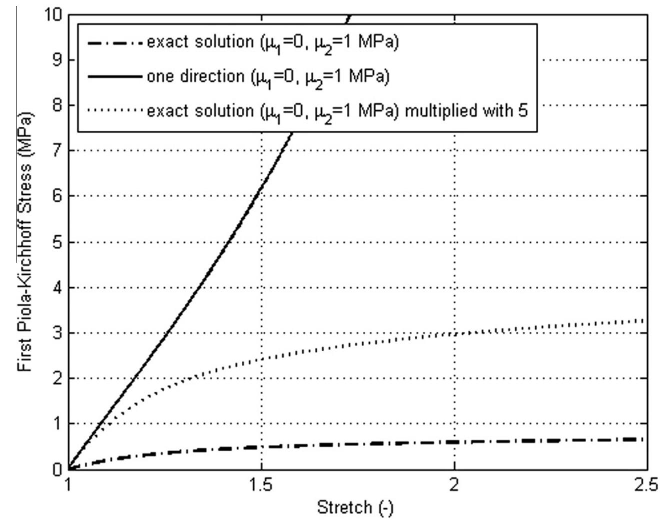

Fig. 5. Stress–stretch responses for $\mu_1 = 0,\ \mu_2 = 1$ MPa under equibiaxial tension.

$$\sigma_{R,\text{uni}}(\lambda) = \rho_R \frac{\mathrm{d}\psi_{\text{iso,uni}}(\lambda)}{\mathrm{d}\lambda} \quad \text{and} \quad \sigma_{R,\text{eb}}(\lambda) = \frac{1}{2}\rho_R \frac{\mathrm{d}\psi_{\text{iso,eb}}(\lambda)}{\mathrm{d}\lambda} \tag{91}$$

If the chain rule of differentiation is applied, an important relation is obtained:

$$\sigma_{R,\text{eb}}(\lambda) = \frac{1}{2}\rho_R \frac{\mathrm{d}\psi_{\text{iso,eb}}(\lambda)}{\mathrm{d}\lambda} = \frac{1}{2}\rho_R(-2\lambda^{-3})\frac{\mathrm{d}\psi_{\text{iso,uni}}(\lambda^{-2})}{\mathrm{d}\lambda^{-2}} = -\lambda^{-3}\sigma_{R,\text{uni}}(\lambda^{-2}) \tag{92}$$

If (91) and (92) are applied, the main tasks to determine the stress–strain relations are the computation of $\psi_{\text{iso,uni}}(\lambda)$ and the calculation of its derivative with respect to λ. The equibiaxial stress-strain curve can be calculated from the uniaxial one using (92). Fortunately, relations (73) and (86) in combination with (87) can also be derived by this method.

6. Concluding remarks

It was the focus of this investigation to calculate exact solutions for some selected load cases and elastic material behaviour. In the special case of nonlinear elasticity of the Mooney–Rivlin type for the directions in the orientation space, closed form solutions were derived for the uniaxial stress-strain behaviour under tension/compression as well as for the equibiaxial behaviour under tension. These solutions exhibit a quite complicated mathematical structure because they consist of combinations of fractional rational functions and transcendental functions. In Carol et al. (2004), different nonlinear elasticity relations were assumed for the directions in the orientation space such that the resulting macroscopic model is of the Mooney–Rivlin type with a compressibility term. A key result of the current study is that the directional approach leads weaker stress responses in comparison with the underlying one-dimensional material model. The intrinsic reason is the averaging process: the average value of the directional

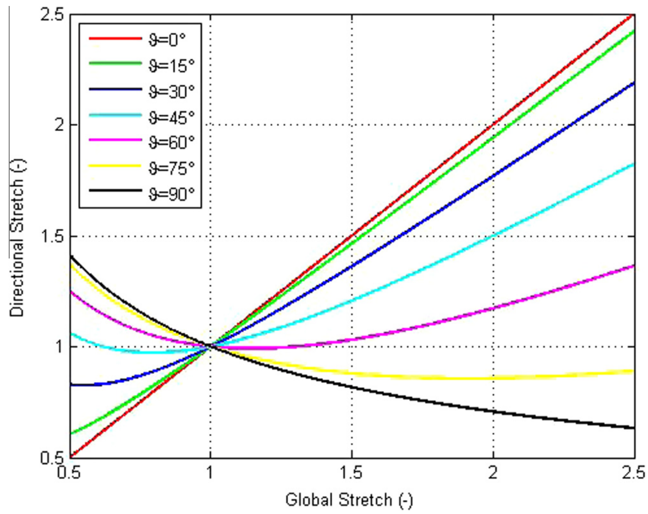

Fig. 6. Directional stretches as function of the global stretch for different angles.

stretch is much smaller than the stretch which is used to evaluate the underlying one-dimensional model. This behaviour is shown in Fig. 6 in which relation (70) is plotted for different values of the angle ϑ. It should be remarked, that this property can be changed or weakened when non-affinity in combination with the nonlinear p-root averaging operator is assumed as proposed by Miehe et al. (2004).

Besides this, the graphical comparisons of Section 5, especially Fig. 5, illustrate the following statement: even if a certain three-dimensional model (e.g. Mooney–Rivlin) is able to fit given experimental data, there is no necessity that a directional law which makes use of the uniaxial response of this three-dimensional model will also be able to fit the same data. Therefore, when developing a directional law, it is not recommended to re-use the uniaxial response of existing three-dimensional models. It seems much wiser to use directional equations which are specifically designed for the occasion that is "tailor-made equations". Nevertheless, the exact solutions which were derived in this paper can be used as references for numerical simulations when the integration over the unit sphere is carried out numerically.

The directional approach can also be applied to represent material behaviour with intrinsic anisotropies. To this end, the material parameters of the uniaxial directional model have to depend on the angles ϑ and φ. This method is investigated in a current research project.

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